Received: *April 15, 2000,* Accepted: *May 28, 2000*

# Wiggly $n$-dimensional object embedded in $(D+1)$-dimensional spacetime

**Jooyoo Hong***

*Department of Physics, Hanyang University at Ansan*
*Ansan, Kyonggi-Do 425-791 Korea*
*E-mail:* jhong@hepth.hanyang.ac.kr

Abstract: Small-scale structures put on an $n$-dimensional object embedded in $(D+1)$-dimensional spacetime modify in general the energy per unit $n$-volume and tension (or pressure) of the coarse-grained $n$-dimensional object. We seek fixed points of the renormalization group equations, i.e. equations of states that are independent of the presence of small-scale structures and find that these fixed points exist only for $n = 1$ case (the string), especially for the Nambu-Goto strings.

Keywords: Field Theories in Higher Dimensions, Brane Dynamics in Gauge Theories, Cosmological Phase Transitions.

*The author wishes to acknowledge the financial support of Hanyang University, Korea, made in the program year of 1998.

Topological defects are a much-studied subject in condensed matter physics. Vortices in superconductor and superfluid systems, textures in $^3$He, line defects/dislocations in crystalline systems and liquid crystals, and most notably the domain wall structures of ferromagnetic materials are several of these examples. The similar topological objects appear also in cosmological contexts, which are probably of more interest to particle physicists. These defects are formed during phase transitions [1], which are closely associated with some form of spontaneous symmetry breaking. According to the grand unification idea the symmetry of the underlying theory, $G$, breaks spontaneously down to smaller symmetries as the universe cools due to its expansion:

$$G \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \longrightarrow \mathrm{SU}(3)_c \times \mathrm{U}(1)_{em} \,.$$

Various models were proposed to predict different types of topological defects at a variety of energy scales, thus leading to a horde of cosmological scenarios and possible astrophysical consequences. Among these topological defects, domain walls and monopoles were known to be of cosmological disaster unless they were formed very recently. Cosmic strings, however, did not encounter such difficulties and, in some scenarios, are to provide a viable fluctuation seed for galaxy formation.

We will investigate the origin of this distinguished role played by cosmic strings among various topological objects by studying the macroscopic motions of these objects and focusing on the equations of state for them. The small-scale structures created by the interactions of these objects with the surroundings or with themselves modify, in the coarse-grained picture, the dynamical properties like the energy per unit volume and tension (or pressure) in a significant way [2, 3]. In some examples, however, the presence of these small-scale structures does not affect the form of equation of state. The existence of this small-scale-structure-independent equation of state is thought to be an important factor in the scale invariance of the energy density of this cosmological object.

For generality we will consider an $n$-dimensional *isotropic* objects embedded in $(D+1)$-dimensional spacetime. The stress-energy-momentum tensor of this object is localized on an $n$-dimensional subspace of $D$-dimensional space. Let $X^\mu(\xi)$ be the location of the world volume with respect to an inertial frame. $\xi = (\xi^0, \xi^1, \ldots, \xi^n)$ are arbitrarily chosen coordinates parametrizing points in the world volume. The $(n+1)$-dimensional metric in the world volume induced by the space-time trajectory of this object is

$$h_{ab} = \partial_a X^\mu \, \partial_b X_\mu \,, \quad a, b = 0, 1, \ldots, n \,, \quad \mu = 0, 1, \ldots, D \,. \tag{1}$$

Here we assume the space-time is flat, i.e. its metric is $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, \ldots, -1)$. At point $X^\mu(\xi)$ on the object resides a stress-energy-momentum tensor describing an $n$-dimensional ideal fluid

$$t^{ab} = (\epsilon - \tau) \, u^a u^b + \tau h^{ab} \,, \tag{2}$$

where $\epsilon$ is the energy per unit $n$-volume of the object, $\tau$ its tension, and $u^a(\xi)$ is the $(n+1)$-velocity of the fluid particle moving parallel to the object thus satisfying $u^a u^b h_{ab} = 1$. The $(D+1)$-dimensional stress-energy-momentum tensor of the $n$-dimensional object is given by

$$T^{\mu\nu}(x) = \int d^{n+1}\xi \sqrt{|h|}\, t^{ab}(\xi)\, \partial_a X^\mu(\xi)\, \partial_b X^\nu(\xi)\, \delta^{D+1}(x - X(\xi))\,, \tag{3}$$

where $h \equiv \det(h_{ab})$. As required, this expression is both invariant under $(n+1)$-dimensional reparametrizations and covariant under $(D+1)$-dimensional Lorentz transformations. Since the motion of the object should conserve $(D+1)$-dimensional energy and momentum, i.e. $\partial_\mu T^{\mu\nu}(x) = 0$, we can easily derive the equations of motion for a general isotropic $n$-dimensional object

$$\partial_a \left[\sqrt{h}\, t^{ab}\, \partial_b X^\mu\right] = 0\,, \quad (\mu = 0, 1, \ldots, D)\,. \tag{4}$$

In addition to the above equations of motion we need an equation of state to have a complete description of dynamics:

$$\tau = \tau(\epsilon)\,. \tag{5}$$

Now we have $D+2$ equations (eqs. (4) and (5)) for $(D+2)$-dynamical variables $\mathrm{X}_\perp(\xi)$, $\epsilon(\xi)$, $\tau(\xi)$, and $\vec{\beta}(\xi) \equiv (u^1(\xi)/u^0(\xi), \ldots, u^n(\xi)/u^0(\xi))$. $\mathrm{X}_\perp(\xi)$ denotes the $(D-n)$-transverse degree of freedom and $\vec{\beta}(\xi)$ is the longitudinal velocity having $n$-independent components.

Now, let us figure out how small-scale structures on the object renormalize its tension and energy per unit $n$-volume. Consider an object lying on average along the $n$-dimensional subspace of $D$-dimensional target space. We use the gauge freedom $\xi^a \to \tilde{\xi}^a(\xi)$ to fix the parameter $\xi^a$ as $(\xi^0, \xi_1, \ldots, \xi_n) = (t, x_1, \ldots, x_n)$. Thus $X^\mu(\xi) = (t, x_1, \ldots x_n, y_1, \ldots, y_{D-n})$. In this gauge the corresponding equations of motion are

$$\partial_a \left(\sqrt{h}\, t^{ab}\right) = 0, \qquad t^{ab}\, \partial_a\, \partial_b\, z = 0\,. \tag{6}$$

An object at rest ($\vec{\beta} = y_1 = \cdots = y_{D-n} = 0$) is characterized by energy per unit $n$-volume $\epsilon_0$ and tension $\tau_0 = \tau(\epsilon_0)$. We first discuss transverse wiggles. At $t = 0$ we perturb the $n$-dimensional object by giving a transverse velocity configuration:

$$y_I^{(0)} = y_I^{(0)}(0, x_1, \ldots,\ x_n)\,, \quad (I = 1, \ldots, D-n)\,, \tag{7}$$

where the index $(a)$ means the differentiation with respect to the coordinate $x_a$. And we set $y_I(0, x_1, \ldots, x_n) = y_I^{(1)}(0, x_1, \ldots, x_n) = \cdots = y_I^{(n)}(0, x_1, \ldots .x_n) = 0$, and the other initial conditions are $\vec{\beta}(0, x_1, \ldots, x_n) = 0$, $\epsilon(0, x_1, \ldots, x_n) = \epsilon_0$ and

JHEP05(2000)049

$\tau(0, x_1, \ldots, x_n) = \tau_0 = \tau(\epsilon_0)$. With these initial conditions and assuming the perturbation is small, we expand eq. (6) in powers of $y^{(0)}$. Describing the perturbed $n$-dimensional object by

$$\begin{aligned}
X^\mu(\xi) &= \Big(t, x_1, \ldots, x_n, y_1(t, x_1, \ldots, x_n), \ldots, y_{D-n}(t, x_1, \ldots, x_n)\Big)\,, \\
\epsilon(t, x_1, \ldots, x_n) &= \epsilon_0 + \epsilon_1(t, x_1, \ldots, x_n)\,, \\
\tau(t, x_1, \ldots, x_n) &= \tau_0 - v_L^2(\epsilon_0)\, \epsilon_1(t, x_1, \ldots, x_n) + O\left(\epsilon_1^2\right), \\
\vec{\beta}(t, x_1, \ldots, x_n) &= \Big(\beta_1(t, x_1, \ldots, x_n), \ldots, \beta_n(t, x_1, \ldots, x_n)\Big)\,,
\end{aligned} \tag{8}$$

where $v_L^2(\epsilon_0) = -\frac{d\tau}{d\epsilon}(\epsilon_0)$, the induced metric and the stress-energy-momentum tensor in the $(n+1)$-dimensional world volume become:

$$\begin{aligned}
h_{ab} &= \eta_{ab} - \sum_I y_I^{(a)} y_I^{(b)}\,, \\
\sqrt{|h|}\, t^{00} &\simeq \epsilon_0 + \epsilon_1 + \frac{\epsilon}{2} \sum_{a,I} y_I^{(a)} y_I^{(a)}\,, \\
\sqrt{|h|}\, t^{0i} &\simeq (\epsilon_0 - \tau_0)\beta_i - \tau_0 \sum_I y_I^{(0)} y_i^{(i)}\,, \\
\sqrt{|h|}\, t^{ii} &\simeq -\tau_0 + v_L^2(\epsilon_0)\, \epsilon_1 - \frac{\tau_0}{2} \sum_{a,I} y_I^{(a)} y_I^{(a)} + \tau_0 \sum_I \left(y_I^{(0)} y_I^{(0)} + y_I^{(i)} y_I^{(i)}\right).
\end{aligned} \tag{9}$$

And to lowest order the relevant equations of motion are written as

$$\begin{aligned}
0 &= \epsilon_0\, y_I^{(00)} - \tau_0 \sum_{i=1}^n y_I^{(ii)}\,, \\
0 &= \partial_t \left[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)}\right] + \vec{\nabla} \cdot \left[(\epsilon_0 - \tau_0)\vec{\beta}\right].
\end{aligned} \tag{10}$$

We observe that the transverse wiggles obey the wave equations and that these transverse oscillations unavoidably generates the longitudinal fluctuations, say $\epsilon_1$ and $\vec{\beta}$. Averaging over space, eq. (10) becomes

$$\frac{d}{dt} \left\langle \epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \right\rangle_{\text{space}} = 0 \tag{11}$$

which in turn implies

$$\left\langle \epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \right\rangle = \left\langle \epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \right\rangle_{\text{space}} \Bigg|_{t=0} = 0$$



JHEP05(2000)049

since $\epsilon_1(0,x_1,\ldots,x_n) = y_I^{(i)}(0,x_1,\ldots,x_n) = 0$. We thus obtain

$$\langle\epsilon_1\rangle = -\frac{1}{2}(\epsilon_0-\tau_0)\left\langle\sum_{i,I} y_I^{(i)}y_I^{(i)}\right\rangle = -\frac{1}{2}(\epsilon_0-\tau_0)\frac{\epsilon_0}{\tau_0}\left\langle\sum_I y_I^{(0)}y_I^{(0)}\right\rangle . \qquad (12)$$

We then proceed to obtain renormalized tension and energy per unit $n$-volume which are defined as follows

$$\langle T_{\mu\nu}(x)\rangle = \mathrm{diag}\left(\bar{\epsilon}_T, -\bar{\tau}_T, \ldots, -\bar{\tau}_T, 0, \ldots, 0\right)\delta^{D-n}(y) . \qquad (13)$$

The result is

$$\bar{\epsilon}_T = \left\langle\sqrt{h}\, t^{00}\right\rangle = \epsilon_0 + \epsilon_0\left\langle\sum_I y_I^{(0)}y_I^{(0)}\right\rangle \qquad (14)$$

and

$$\begin{aligned}\bar{\tau}_T &= -\left\langle\sqrt{h}t^{11}\right\rangle \\ &= \tau_0 - \frac{1}{2}\left[\tau_0 - \left(1-\frac{2}{n}\right)\epsilon_0 - v_L^2(\epsilon_0)(\epsilon_0-\tau_0)\frac{\epsilon_0}{\tau_0}\right]\left\langle\sum_I y_I^{(0)}y_I^{(0)}\right\rangle , \end{aligned} \qquad (15)$$

where we used eq. (12) and $\langle\sum_I y_I^{(i)}y_I^{(i)}\rangle$ assumed to be independent of the index i to maintain the isotropy of the coarse-grained $n$-dimensional object.

Next, let us consider longitudinal wiggles. We should carefully specify initial conditions since the Lorentz contraction due to initial non-zero longitudinal velocity also changes the initial parametrization made when there are no wiggles, i.e. when $\vec{\beta}(0,x_1,\ldots,x_n) = 0$. We need to change the initial value of energy per unit $n$-volume to make up for the Lorentz contraction effect. If we put small longitudinal wiggles of velocity $\vec{\beta}$, the volume of the object decreases by $\delta V = V_0 - V \simeq \frac{1}{2}\vec{\beta}^2 V$. Then the energy conservation requires the energy per unit $n$-volume to change: $\delta(\epsilon V) = \tau\delta V$, i.e. $\delta\epsilon = -(\epsilon-\tau)\frac{\delta V}{V} = -\frac{1}{2}(\epsilon-\tau)\vec{\beta}^2$.

In the gauge $(\xi^a) = (t,x_1,\ldots,x_n)$ the $n$-dimensional object perturbed by longitudinal wiggles is described by

$$\begin{aligned} X^\mu(\xi) &= (t,x_1,\ldots,x_n,0,\ldots,0) , \\ \epsilon(t,x_1,\ldots,x_n) &= \epsilon_0 + \epsilon_1(t,x_1,\ldots,x_n) + \epsilon_2(t,x_1,\ldots,x_n) + \cdots , \\ \tau(t,x_1,\ldots,x_n) &= \tau_0 + \tau_1(t,x_1,\ldots,x_n) + \tau_2(t,x_1,\ldots,x_n) + \cdots \\ &= \tau_0 - v_L^2(\epsilon_0)\,\epsilon_1(t,x_1,\ldots,x_n) - v_L^2(\epsilon_0)\,\epsilon_2(t,x_1,\ldots,x_n) - \\ &\quad - \frac{1}{2}\frac{dv_L^2}{d\epsilon}(\epsilon_0)\,\epsilon_1^2(t,x_1,\ldots,x_n) + \cdots , \\ \vec{\beta}(t,x_1,\ldots,x_n) &= \vec{\beta}_1(t,x_1,\ldots,x_n) + \vec{\beta}_2(t,x_1,\ldots,x_n) + \cdots . \end{aligned} \qquad (16)$$

For longitudinal wiggles the induced metric is trivial, i.e. $h_{ab} = \eta_{ab} = \mathrm{diag}(+1, -1, \ldots, -1)$. Thus the $(n+1)$-dimensional stress-energy-momentum tensor is written:

$$\begin{aligned}
t^{00} &\simeq \epsilon_0 + \epsilon_1 + \epsilon_2 + (\epsilon_0 - \tau_0)\, \vec{\beta}_1^2 \,, \\
t^{0i} &\simeq (\epsilon_0 - \tau_0)(\beta_{1i} + \beta_{2i}) + \left(1 + v_L^2\,(\epsilon_0)\right) \epsilon_1\, \beta_{1i} \,, \\
t^{ii} &\simeq -\tau_0 + v_L^2\,(\epsilon_0)(\epsilon_1 + \epsilon_2) + v_L\,(\epsilon_0)\, \frac{dv_L}{d\epsilon}\,(\epsilon_0) + (\epsilon_0 - \tau_0)\, \beta_{1i}^2 \,, \\
t^{ij} &\simeq (\epsilon_0 - \tau_0)\, \beta_{1i}\, \beta_{1j} \,.
\end{aligned} \tag{17}$$

Following the same steps taken for the transverse wiggles, we can obtain the expressions for renormalized tension and energy per unit $n$-volume due to the presence of longitudinal wiggles. Namely, longitudinal wiggles renormalize the energy per unit $n$-volume and tension as follows:

$$\begin{aligned}
\bar{\epsilon}_L &= \epsilon_0 + (\epsilon_0 - \tau_0) \left\langle \vec{\beta}_1^2 \right\rangle , \\
\bar{\tau}_L &= \tau_0 - (\epsilon_0 - \tau_0) \left[ \frac{1}{n} + (\epsilon_0 - \tau_0)\, \frac{d \ln v_L}{d\epsilon}\,(\epsilon_0) \right] \left\langle \vec{\beta}_1^2 \right\rangle .
\end{aligned} \tag{18}$$

If there are transverse and longitudinal wiggles with various wavelengths whose spectral densities of $\langle \dot{z}^2 \rangle$ and $\langle \vec{\beta}_1^2 \rangle$ are denoted by $W_T(k)$ and $W_L(k)$, respectively on a $\ln k$ scale, the renormalization group equations for $\epsilon\,(k)$ and $\tau(k)$ are

$$\begin{aligned}
-\frac{d\epsilon}{d\ln k} &= W_T(k)\epsilon + W_L(k)(\epsilon - \tau) \,, \\
-\frac{d\tau}{d\ln k} &= -\frac{1}{2}\, W_T(k) \left[ \tau + \frac{2-n}{n}\, \epsilon - v_L^2\, \epsilon \left( \frac{\epsilon}{\tau} - 1 \right) \right] - \\
&\quad - W_L(k)(\epsilon - \tau) \left[ \frac{1}{n} + (\epsilon - \tau)\, \frac{d \ln v_L}{d\epsilon} \right] .
\end{aligned} \tag{19}$$

Fixed points of these equations depend upon the ratio $W_T/W_L$ in general. There may exist, however, equations of state invariant under the change of wiggle spectrum. They should be, if any, the common fixed points of the transverse RG equation and the longitudinal RG equation, i.e. the RG equation for the transverse wiggles only and that for the longitudinal wiggles only, respectively. The transverse RG equation has the following fixed points

$$\frac{\epsilon\, \tau^n}{(\epsilon + n\, \tau)^{n-1}} = \text{constant} \tag{20}$$

which do not change under addition or subtraction of transverse wiggles. And if we integrate the longitudinal RG equation, we arrive at the following relation:

$$\frac{(\epsilon - \tau)^2\, \left| \frac{d\tau}{d\epsilon} \right|^n}{\left(1 + n\, \frac{d\tau}{d\epsilon}\right)^{n+1}} = \text{constant}\,. \tag{21}$$

However, eq. (20) does not satisfy eq. (21) for arbitrary $n$. Only for $n = 1$ (the string) does there exist a fixed point that satisfies both transverse and longitudinal RG equations. This particular equation of state is

$$\epsilon\,\tau = \text{constant} \tag{22}$$

to which class the Nambu-Goto strings belong. Thus, equations of state independent of wiggle spectra exist only for Nambu-Goto strings, which may explain the unique status of cosmic strings compared to other higher-dimensional objects. We believe that the existence of this wiggle-independent equation of state is closely linked with the scale invariant spectra of the cosmic string networks[2].

## Acknowledgments

The author thanks J. Kim and P. Sikivie for helpful discussions.

## References

[1] For a thorough review on this subject, see A. Vilenkin and E.P.S. Shellard, *Cosmic strings and other topological defects*, Cambridge University Press, Cambridge 1994.

[2] J. Hong, J. Kim and P. Sikivie, *Wiggly relativistic strings*, *Phys. Rev. Lett.* **69** (1992) 2611 [hep-ph/9210210];
J. Kim and P. Sikivie, *Stretching Wiggly strings*, *Phys. Rev.* **D 50** (1994) 7410 [hep-ph/9405227].

[3] J. Hong, *Wiggly relativistic membranes*, *Phys. Rev.* **D 50** (1994) 5435.

[4] A. Vilenkin, *Effect of small scale structure on the dynamics of cosmic strings*, *Phys. Rev.* **D 41** (1990) 3038.